Numerical Solution of Cauchy Problems for Elliptic Equations in “Rectangle-like” Geometries

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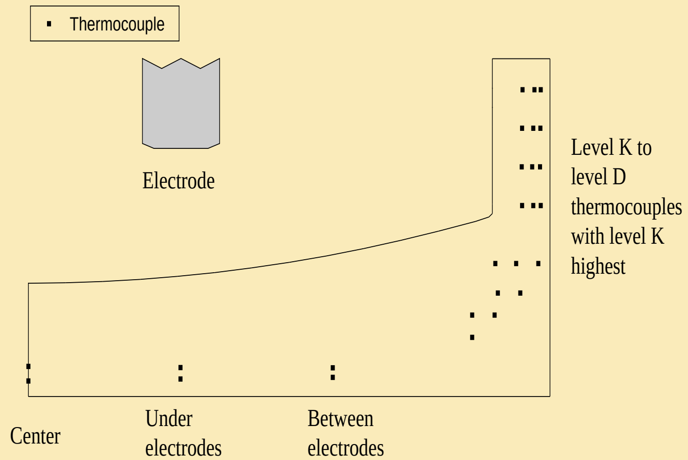
Overview

- Introduction, Motivating Example
- Ill–posedness, Stabilization
- Numerical Test problem in FEMLAB
- Transformation to rectangular geometry. Mapping of Normal derivatives.
- Numerical solution of Test problem
- Conclusions, Future Work

Motivating example: Ilmenite iron melting furnace



The furnace material properties are temperature dependent.

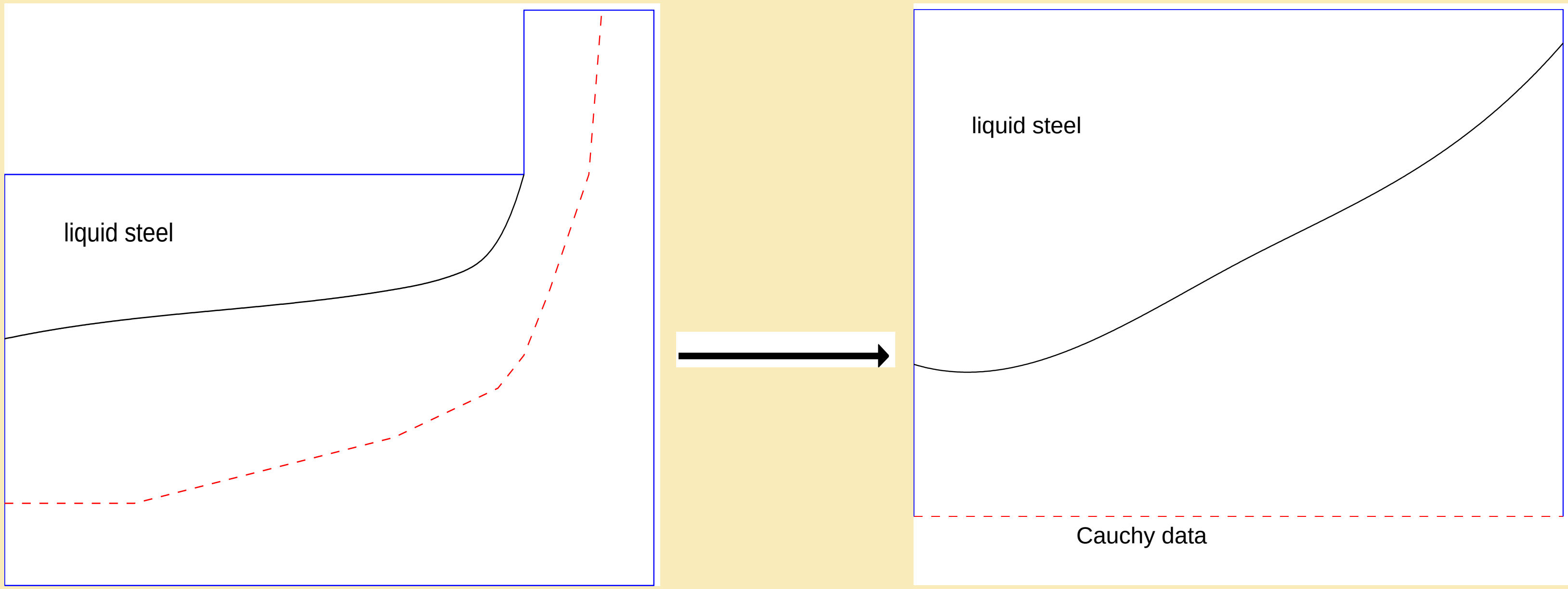
Problem: find the inner shape of the furnace.

Nonlinear, and (rather) complex geometry

PhD thesis: I M Skaar, Monitoring the Lining of a Melting Furnace, NTNU, Trondheim, 2001

Irregular geometry - non-constant coefficient

Map the region to a rectangle:



Find lining: the interface between iron and furnace

The Cauchy Problem for Laplace's equation

Classical ill-posed problem (Hadamard, 1900-1930)

$$
\begin{aligned}
&u_{xx} + u_{yy} = 0, \\
&u(x,0) = 0, \qquad -\infty \leq x \leq \infty, \\
&\frac{\partial u}{\partial x}(x,0) = \frac{1}{n}\sin(nx), \qquad -\infty \leq x \leq \infty.
\end{aligned}
$$

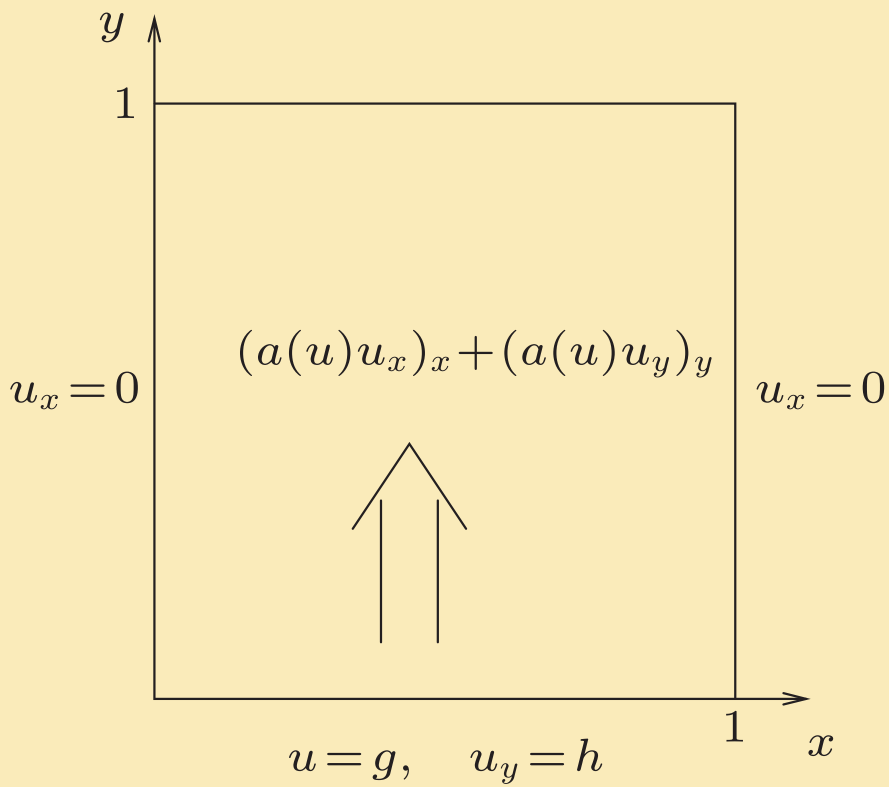
Solution:

$$u(x,y) = \frac{1}{n^2}\sin(nx)\sinh(ny).$$

Large n: arbitrarily small data, arbitrarily large solution.

The solution does not depend continuously on the data!

Ill–posed Cauchy Problem on Unit Square



Fourier analysis

By separation of variables:

$$T(x,y) = A_0 y + B_0 + \sum_{k=1}^{\infty} (A_k \mathrm{e}^{k\pi y} + B_k \mathrm{e}^{-k\pi y}) \cos(k\pi x),$$

Fourier coefficients $\{A_k\}$ and $\{B_k\}$ satisfy (for $k > 0$):

$$\begin{pmatrix} 1 & 1 \\ k\pi & -k\pi \end{pmatrix} \begin{pmatrix} A_k \\ B_k \end{pmatrix} = \begin{pmatrix} \widehat{g}_k \\ \widehat{h}_k \end{pmatrix},$$

where $\{\widehat{g}_k\}$ and $\{\widehat{h}_k\}$ are the Fourier–cosine coefficients of $g(x)$ and $h(x)$

Since $\mathrm{e}^{k\pi y} \to \infty$ as $k \to \infty$ the problem is severely ill–posed!

Rewrite as an Initial value problem:

$$\begin{pmatrix} u \\ au_y \end{pmatrix}_y = \begin{pmatrix} 0 & a^{-1} \\ -\frac{\partial}{\partial x} a \frac{\partial}{\partial x} & 0 \end{pmatrix} \begin{pmatrix} u \\ au_y \end{pmatrix}, \quad 0 \leq y \leq 1, \quad \begin{pmatrix} u(x,0) \\ u_y(x,0) \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}.$$

The unbounded x-derivative makes the problem ill-posed!

Approximation of derivative:

$$\frac{\partial}{\partial x}\left(a(u)\frac{\partial v}{\partial x}\right) \approx \bar{D}(A\bar{D}V)$$

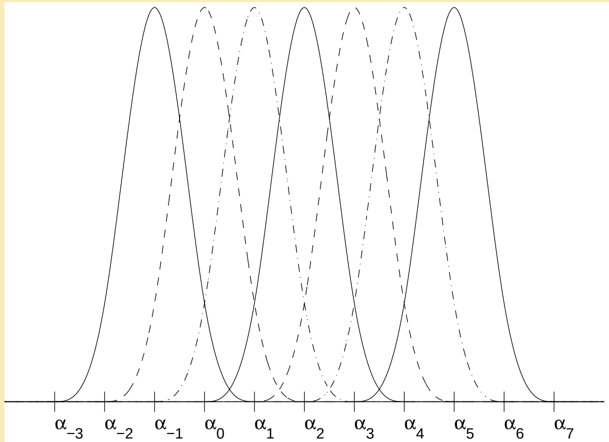
A diagonal matrix, depends on u

$\bar{D}$ bounded differentiation matrix, spectral or otherwise

Resulting stable problem is solved using standard code (`ode45`) in Matlab.

Approximate $u(x)$ by a least squares cubic spline $s(x)$.

Then set $u'(x) \approx s'(x)$.

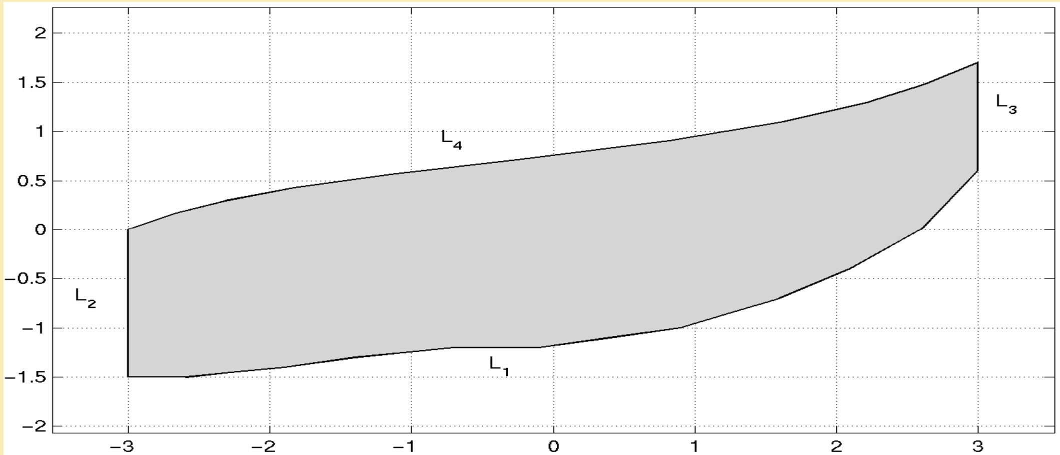


The basis functions $B_j^3(x)$, for $j = -1, \ldots, 5$.

Choose a coarse grid $\Longrightarrow$ bounded differentiation operator.

Advantage: flexible at boundaries.

A Numerical Model Problem



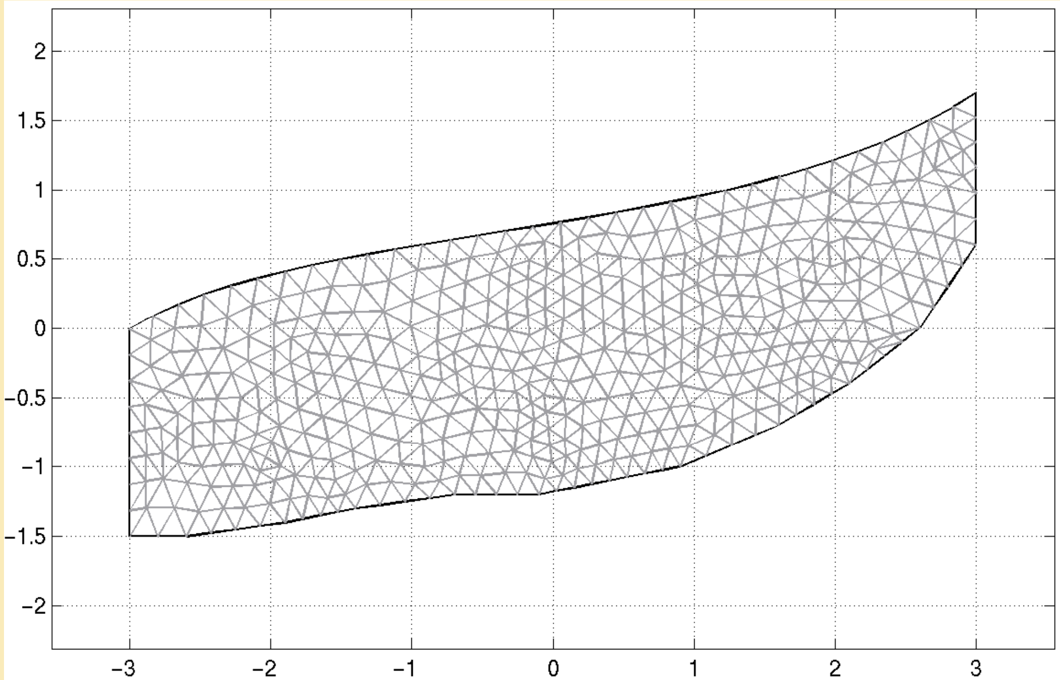
The Heat Equation

$$(a(u)u_x)_x + (a(u)u_y)_y = 0, \text{ in } \Omega$$

Boundary Conditions

$$u = g, \quad \frac{\partial u}{\partial n} = h, \quad \text{on} \quad L_1,$$
$$\frac{\partial u}{\partial n} = 0, \quad \text{on} \quad L_2 \quad \text{and} \quad L_3,$$

Test problem generated in FEMLAB



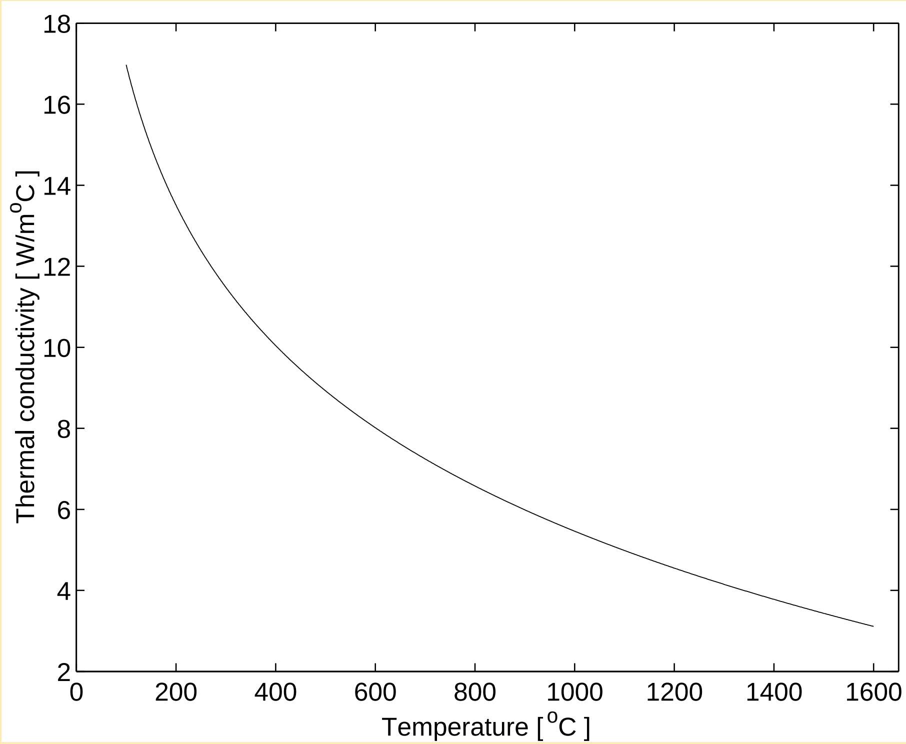
Sides: insulated

Upper boundary: 1450^o

Lower boundary: 800^o

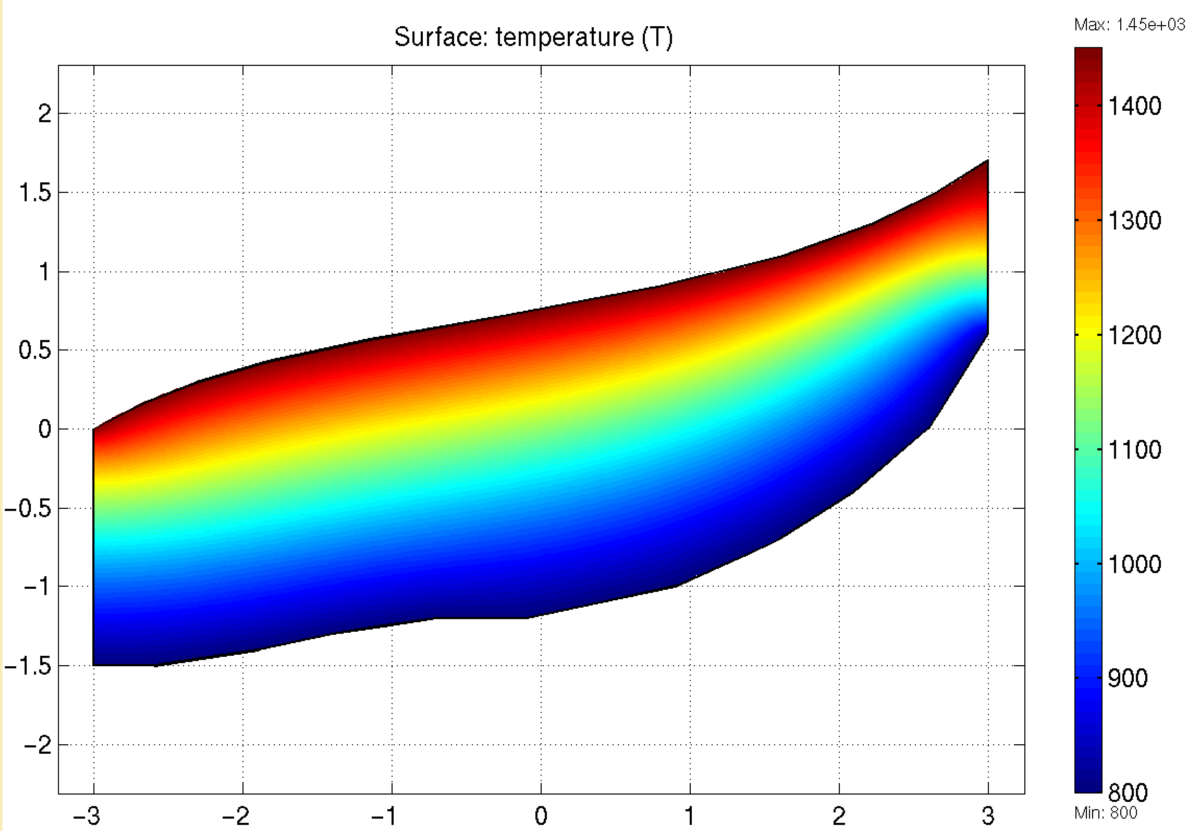
Temperature-dependent conductivity

Thermal conductivity



The thermal conductivity for magnesia brick, as a function of temperature.

Constructed solution (direct problem solved using FEMLAB):



Temperature and Heat-Flux data along the lower boundary is used to identify the upper boundary!

Transformation of the Problem

The Conformal Mapping

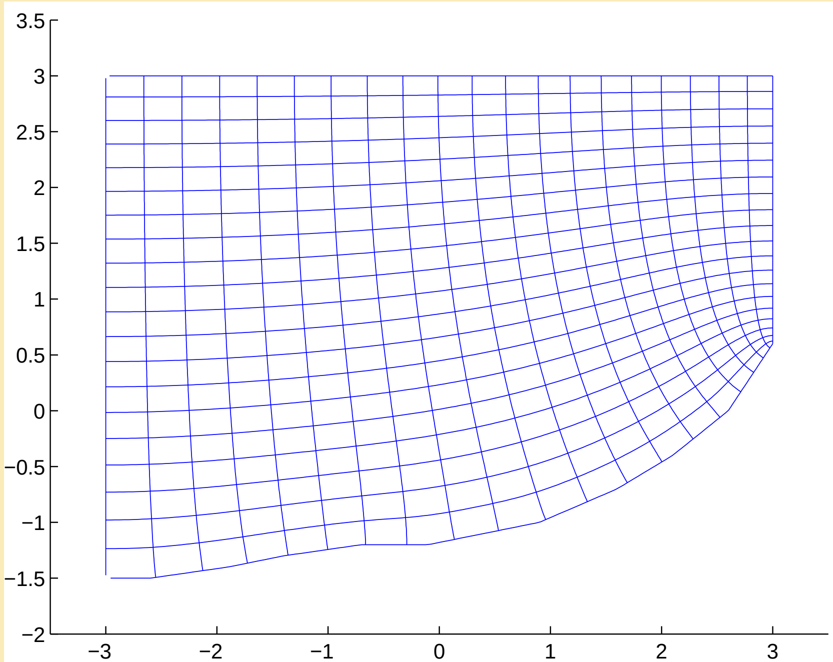
$$(x, y) = \phi(\xi, \eta) = [x(\xi, \eta), y(\xi, \eta)]$$

The problem is transformed into

$$\begin{cases} (a(v)v_\xi)_\xi + (a(v)v_\eta)_\eta = 0, & 0<\xi<1, 0<\eta<1 \\ v(\xi, 0) = g(\phi(\xi, 0)), & 0<\xi<1, \\ v_\eta(\xi, 0) = |\phi'|h(\phi(\xi, 0)), & 0<\xi<1, \\ v_\xi(0, \eta)=v_\xi(1, \eta)=0, & 0<\eta<L_\eta, \end{cases}$$

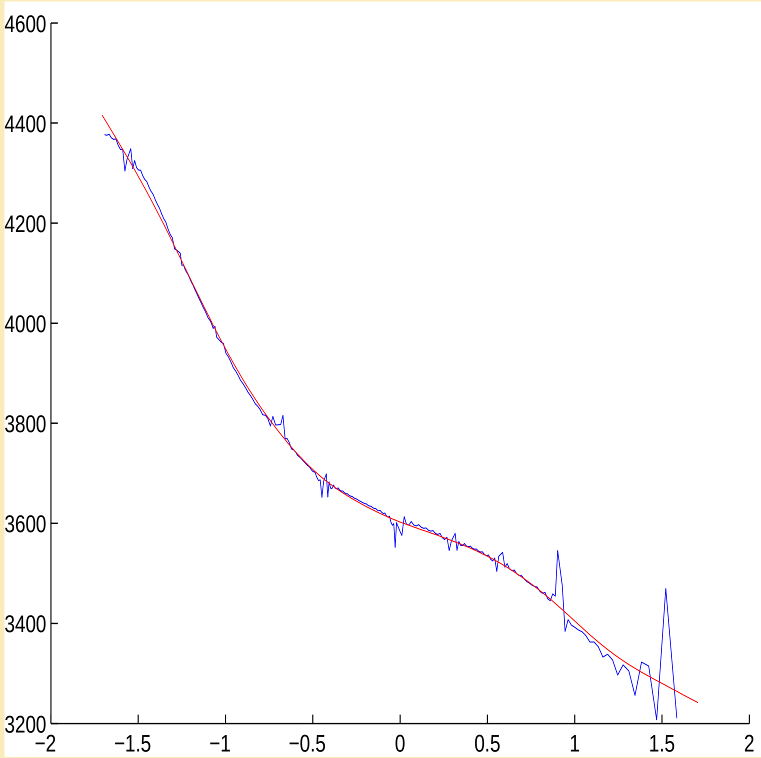
Schwarz-Christoffel Matlab toolbox: polygonal regions

Transformation of the area



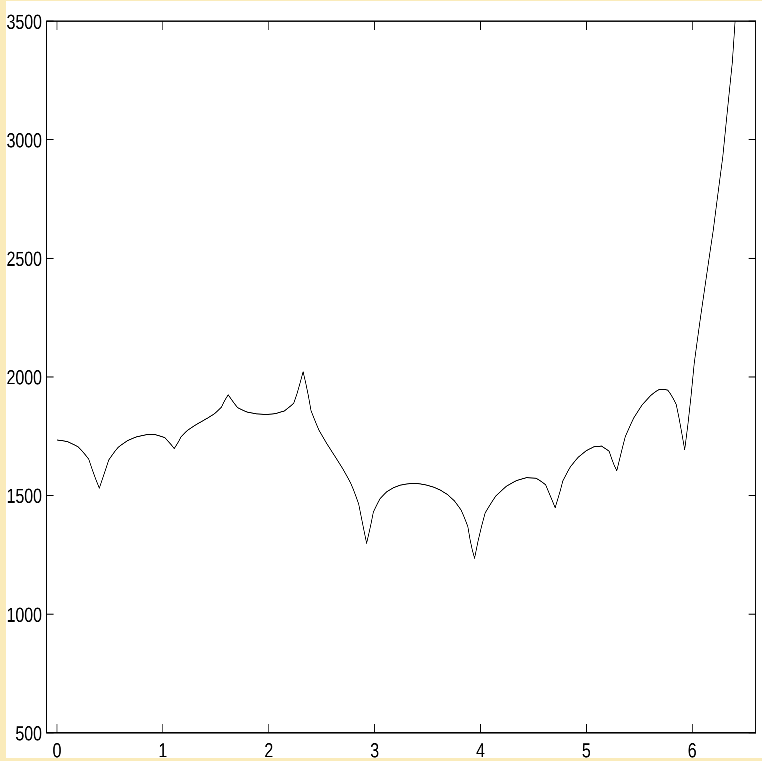
Conformal mapping of the auxilliary domain onto a rectangle. Actual grid is finer!

Computed flux data on the square



Transformed using Conformal mapping (blue curve). No noise added!

Computed heat-flux on original Domain



The Heat-Flux $\dfrac{\partial u}{\partial n}$ on L_1 computed by FEMLAB

Transformation of normal derivatives

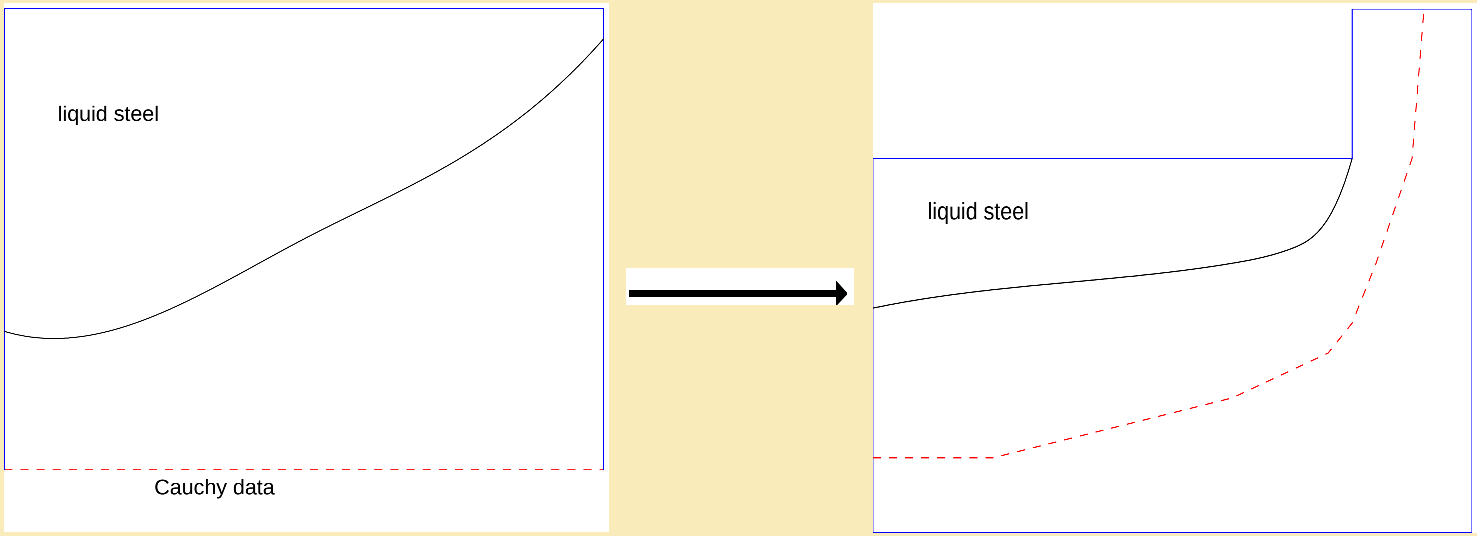
By the Conformal mapping

$$\frac{\partial u}{\partial n}|_{L1} = |(\phi^{-1})'| \frac{\partial v}{\partial \eta}(\xi, 0)$$

The Schwarz–Christoffel mapping function ϕ has a singularity at every corner of the polygonal domain.

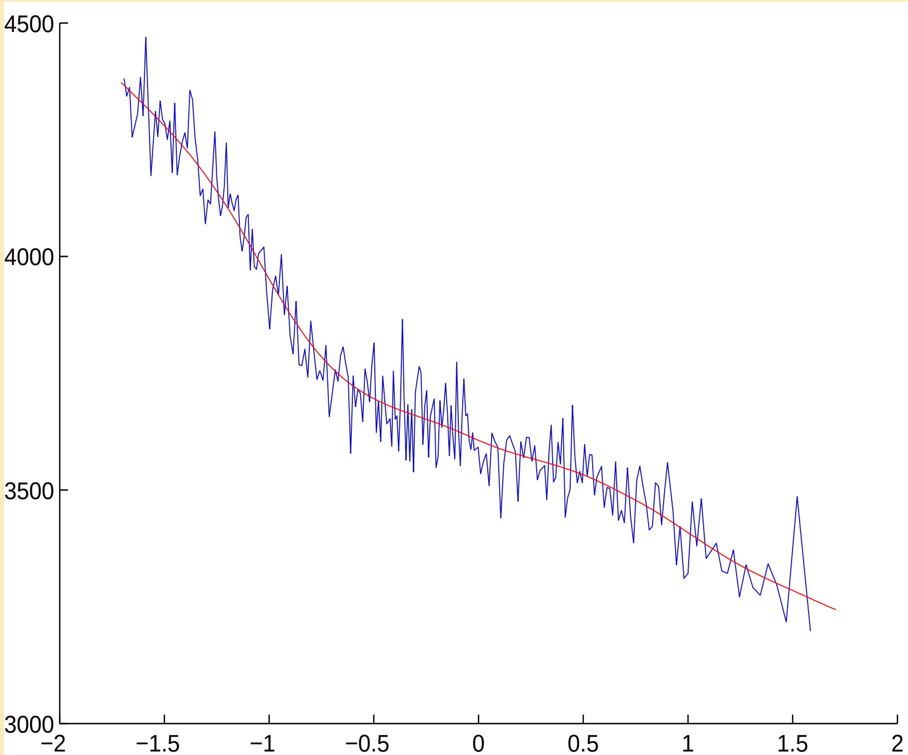
The Singularities in $|(\phi^{-1})'|$ almost cancel out the spikes in FEMLABs normal derivative.

Numerical Solution of the Test problem



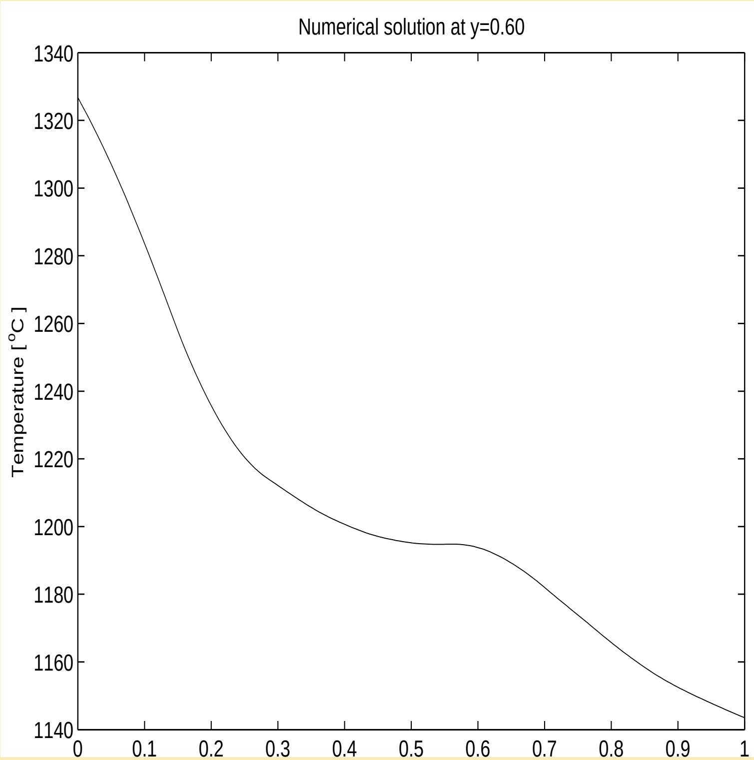
- Solve Cauchy problem. Find the isotherm $v = 1450\,^{o}C$.
- Conformal mapping of the identified curve.

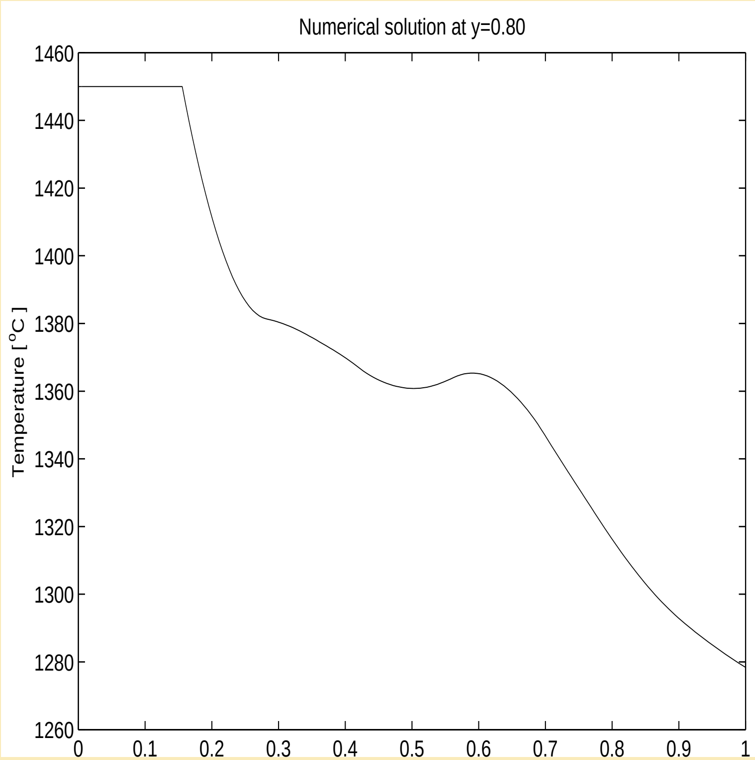
Noisy flux data



Normally distributed noise added. Realistic noise level!

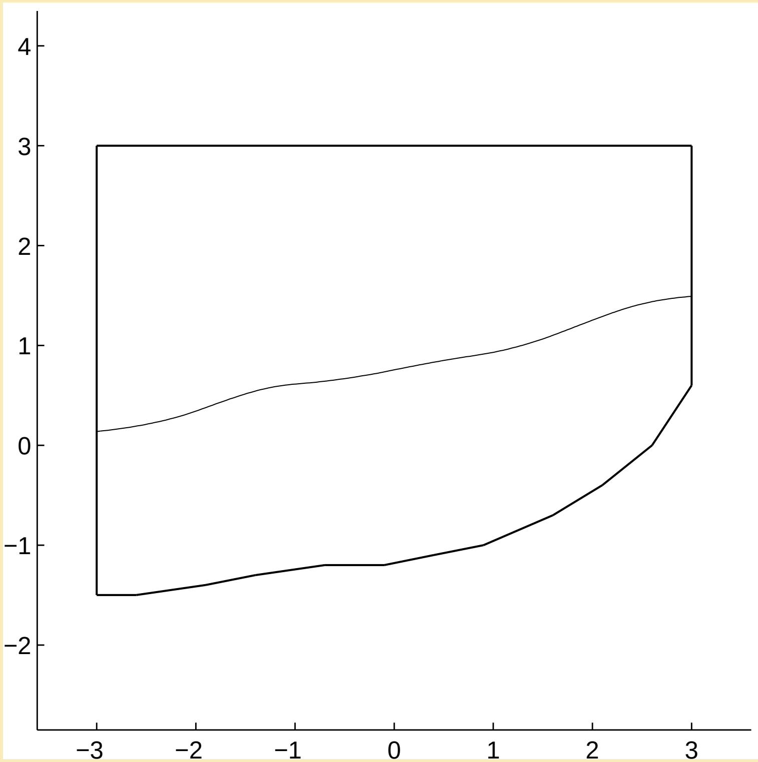
Computed temperatures at 0.6 and 0.8

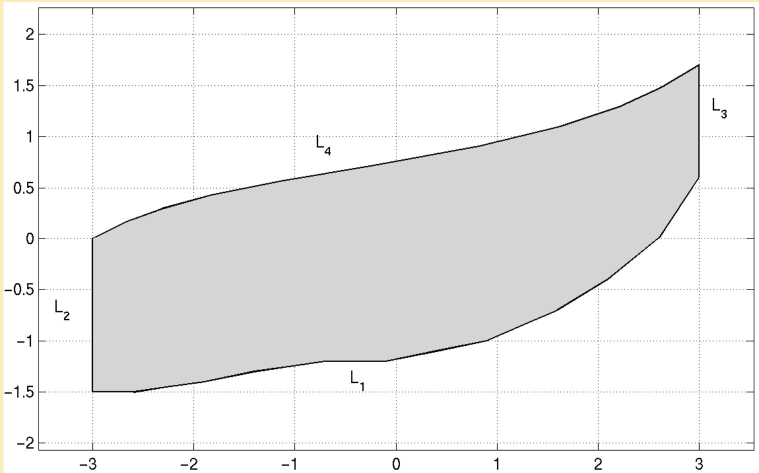




Derivatives computed using 11 B-splines. Differential equation only valid when temperature is below $1450\,^oC$.

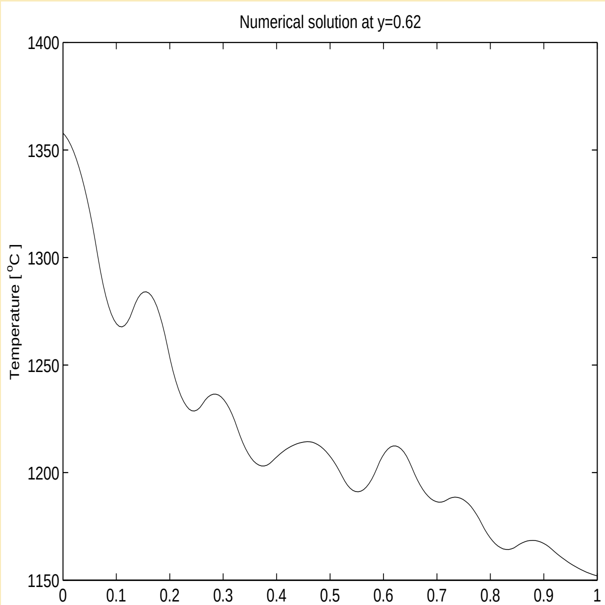
Noisy data, identified boundary

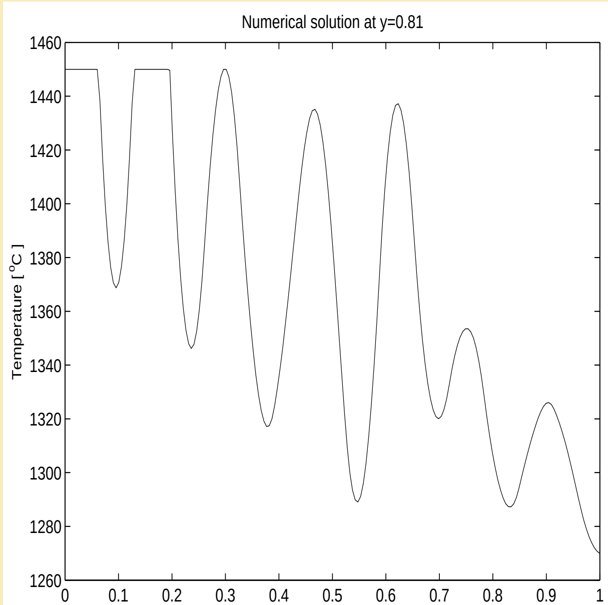




Derivatives computed using 11 B-splines.

Less regularization, temperature at 0.62 and 0.81





Derivatives computed using 19 B-splines!

> Impossible to identify boundary when the temperature is not smooth

Conclusions

- Efficient solution: replace a derivative by bounded approximation, solve Cauchy problem stably using a "method of lines"

- General geometries: transform "rectangle-like region" to rectangle (equivalent to creating a curvilinear mesh). Solve the problem on the rectangle

- Transformation cheap to compute for rather general geometries (approximated by polygon)

- Nonlinear problems can be solved rather easily

- Stability theory: *A stability estimate for a Cauchy problem for an elliptic partial differential equation*, Inverse Problems, vol. 21, no 5, pp. 1643-1653, October 2005.

Future Work

- More detailed FEMLAB model of the Furnace. More realistic simulation of measurements.
- Identify boundaries given other conditions (e.g. insulated boundary).
- Applications: The Melting Furnace
- Use FEMLAB to creat Test problem. Finding good test problems is very important!