Testing Equality of Scale Parameters of Two Weibull Distributions in the Presence of Unequal Shape Parameters

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August 15, 2014

Outline

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1. Introduction and Motivating Examples

- Data in the form of survival times arise in many fields of studies such as engineering, manufacturing, aeronautics and bio-medical sciences.
- A popular model for survival data is the two parameter Weibull distribution.
- Let Y be a random variable that follows a two parameter Weibull distribution with shape parameter β and scale parameter α. Then the probability density function of Y can be written as

$$f(y) = \frac{\beta}{\alpha}\left(\frac{y}{\alpha}\right)^{(\beta-1)} exp\left[-\left(\frac{y}{\alpha}\right)^{\beta}\right]; \;\; y \geq 0; \; \beta, \; \alpha > 0.$$

1. Introduction and Motivating Examples

- Often lifetime or survival time data that are collected in the form of two independent samples are assumed to have come from two independent Weibull populations with different shape and scale parameters.
- In such a situation it may be of interest to test the equality of the scale parameters with the shape parameters being unspecified.
- Let $y_{11}, y_{12}, \cdots, y_{1n_1}$ and $y_{21}, y_{22}, \cdots, y_{2n_2}$ be samples from two independent Weibull populations with parameters (α_1, β_1) and (α_2, β_2) respectively. Our objective is to test the null hypothesis $H_0 : \alpha_1 = \alpha_2$, where β_1 and β_2 are unspecified.

- This problem is equivalent to testing the equality of the location parameters with the shape parameters being unspecified in two extreme value distributions.
- Also, this is analogous to the traditional Behrens-Fisher problem of testing the equality of the means μ_1 and μ_2 of two normal populations where the variances σ_1^2 and σ_2^2 are unknown.

1. Motivating Example:

Table 1: Failure Times of Different Bearing Specimens

Type of Compound				
I	*II*	*III*	*IV*	*V*
3.03	3.19	3.46	5.88	6.43
5.53	4.26	5.22	6.74	9.97
5.60	4.47	5.69	6.90	10.39
9.30	4.53	6.54	6.98	13.55
9.92	4.67	9.16	7.21	14.45
12.51	4.69	9.40	8.14	14.72
12.95	5.78	10.19	8.59	16.81
15.21	6.79	10.71	9.80	18.39
16.04	9.37	12.58	12.28	20.84
16.84	12.75	13.41	25.46	21.51

1. Motivating Example:

TABLE 2: Estimates of parameters obtained by different methods for compound combinations of bearing specimens data in Table 1

	Compound Type Combinations									
	(I, II)	(I, III)	(I, IV)	(I, V)	(II, III)	(II, IV)	(II, V)	(III, IV)	(III, V)	(IV, V)
$\hat{\alpha}_0$	9.0056	10.4848	11.7213	14.7887	8.5093	7.1645	9.5075	8.6567	13.8549	15.0676
$\hat{\beta}_{10}$	1.8385	2.2491	2.5351	2.4628	2.3276	2.3718	2.1549	2.3160	2.2909	1.9228
$\hat{\beta}_{20}$	2.2376	3.2077	1.9758	3.1844	2.6780	1.5713	1.4804	1.3236	2.8340	3.2844
$\hat{\alpha}_{1a}$	12.0607	12.0607	12.0607	12.0607	6.8596	6.8596	6.8596	9.6847	9.6847	7.5107
$\hat{\alpha}_{2a}$	6.8596	9.6847	7.5107	16.3507	9.6847	7.5107	16.3507	7.5107	16.3507	16.3507
$\hat{\beta}_{1a}$	2.5881	2.5881	2.5881	2.5881	2.3202	2.3202	2.3202	3.1324	3.1324	4.0912
$\hat{\beta}_{2a}$	2.3202	3.1324	4.0912	3.6518	3.1324	4.0912	3.6518	4.0912	3.6518	3.6518
$\hat{\alpha}_{cr}$	8.2096	10.4267	11.6395	14.1747	8.0635	7.6565	9.3231	10.0022	11.7626	14.1524
$\hat{\beta}_{1cr}$	2.4941	2.4941	2.4941	2.4941	2.6956	2.6956	2.6956	3.0152	3.0152	2.5244
$\hat{\beta}_{2cr}$	2.6956	3.0152	2.5244	3.5348	3.0152	2.5244	3.5348	2.5244	3.5348	3.5348
$\hat{\alpha}_{tg}$	7.9949	9.3692	11.7085	12.7716	7.7511	8.2172	9.0596	9.7884	10.6421	14.1160
$\hat{\beta}_{1tg}$	2.0733	2.0733	2.0733	2.0733	2.2457	2.2457	2.2457	2.5192	2.5192	2.0992
$\hat{\beta}_{2tg}$	2.2457	2.5192	2.0992	2.9636	2.5192	2.0992	2.9636	2.0992	2.9636	2.9636

1. Introduction and Motivating Examples

- We develop four test procedures, namely, a likelihood ratio test,
- a $C(\alpha)$ test based on the maximum likelihood estimates of the nuisance parameters,
- a $C(\alpha)$ test based on the method of moments estimates of the nuisance parameters by Cran (1988)
- and a $C(\alpha)$ test based on the method of moments estimates of the nuisance parameters by Teimouri and Gupta (2013).

2. Estimates of the Parameters: Maximum Likelihood

- the maximum likelihood estimates of the parameters α_i, β_i, under the alternative hypothesis are obtained by solving the estimating equations

$$-\frac{n_i\beta_i}{\alpha_i} + \frac{\beta_i}{\alpha_i^{\beta_i+1}} \sum_{j=1}^{n_i} y_{ij}^{\beta_i} = 0$$

and

$$\frac{n_i}{\beta_i} + \sum_{j=1}^{n_i} \log\left(y_{ij}\right) - n_i \log\left(\alpha_i\right) + \frac{\log(\alpha_i)}{\alpha_i^{\beta i}} \sum_{j=1}^{n_i} y_{ij}^{\beta_i} - \frac{1}{\alpha_i^{\beta_i}} \sum_{j=1}^{n_i} y_{ij}^{\beta_i} \log\left(y_{ij}\right) = 0$$

simultaneously.

2. Estimates of the Parameters: Maximum Likelihood

- the maximum likelihood estimates of the parameters α β_1, β_2, under the null hypothesis are obtained by solving the estimating equations

$$\sum_{i=1}^{2}\left[-\frac{n_i\beta_i}{\alpha}+\frac{\beta_i}{\alpha^{\beta_i+1}}\sum_{j=1}^{n_i} y_{ij}^{\beta_i}\right]=0,$$

$$\frac{n_1}{\beta_1}+\sum_{j=1}^{n_1}\log\left(y_{1j}\right)-n_1\log(\alpha)+\frac{\log(\alpha)}{\alpha^{\beta_1}}\sum_{j=1}^{n_1} y_{1j}^{\beta_1}-$$
$$\frac{1}{\alpha^{\beta_1}}\sum_{j=1}^{n_1} y_{1j}^{\beta_1}\log\left(y_{1j}\right)=0$$

2. Estimates of the Parameters: Maximum Likelihood

- and

$$\frac{n_2}{\beta_2} + \sum_{j=1}^{n_2} \log\left(y_{2j}\right) - n_2 \log(\alpha) + \frac{\log(\alpha)}{\alpha^{\beta_2}} \sum_{j=1}^{n_2} y_{2j}^{\beta_2} - \frac{1}{\alpha^{\beta_2}} \sum_{j=1}^{n_2} y_{2j}^{\beta_2} \log\left(y_{2j}\right) = 0$$

simultaneously. Denote the maximum likelihood estimates of $\delta = (\alpha,\ \beta_1,\ \beta_2)$ by $\hat{\delta}_{ml} = \left(\hat{\alpha}_{ml},\ \hat{\beta}_{1ml},\ \hat{\beta}_{2ml}\right)$.

2. Estimates of the Parameters: Method Moments by Cran (1988)

- Cran (1988) proposes moments estimates of the parameters for the three-parameter Weibull distribution and applies this procedure for the two-parameter model considering the location parameter as zero. Following Cran (1988) the estimates of the parameters α_i and β_i, under the alternative hypothesis, are $\hat{\alpha}_{ic} = \dfrac{\bar{m}_1}{\Gamma\left(1 + \frac{1}{\hat{\beta}_{ic}}\right)}$ and $\hat{\beta}_{ic} = \frac{ln(2)}{ln(\bar{m}_1) - ln(\bar{m}_2)}$, where $\bar{m}_k = \sum_{r=0}^{n-1} \left(1 - \frac{r}{n}\right)^k \left\{y_{(r+1)} - y_{(r)}\right\}$, with $y_{(0)} = 0$ and $y_{(r)}$ is the r^{th} ordered observation.

2. Estimates of the Parameters: Method Moments by Cran (1988)

- Note that the estimate of β_i is independent of α_i, so, it should be the same under the null and the alternative hypotheses.
  As a moment estimate of the common value α of α_1 and α_2 under the null hypothesis we use a weighted average of $\hat{\alpha}_{ic}$ as

  $\hat{\alpha}_c = \sum_{i=1}^{2} w_i \hat{\alpha}_{ic} / \sum_{i=1}^{2} w_i$, where $w_i = \dfrac{n_i}{\hat{V}_{ic}}, \; i = 1, \; 2,$

  $\hat{V}_{ic} = \hat{\alpha}_{ic}^2 \left[\Gamma\left(1 + \dfrac{2}{\hat{\beta}_{ic}}\right) - \left\{ \Gamma\left(1 + \dfrac{1}{\hat{\beta}_{ic}}\right) \right\}^2 \right]$ and V_{ic} is the variance of a random variable from the Weibull (α_i,β_i) population (see Lawless, 1982). Denote these method of moments estimates by $\hat{\delta}_c = \left(\hat{\alpha}_c, \; \hat{\beta}_{1c}, \; \hat{\beta}_{2c}\right)$.

2. Estimates of the Parameters: Method Moments by Teimouri and Gupta (2013)

- In a recent article Teimouri and Gupta (2013) also propose a method of moment estimate of the shape parameter of a three-parameter Weibull distribution and apply this method to a two-parameter Weibull distribution for estimating the shape parameter of a two parameter Weibill distribution. As the estimate by Cran (1988) this estimate is also independent of the estimate of the scale parameter α.

  Following Teimouri and Gupta (2013) the moment estimate of β_i is $\hat{\beta}_{itg} = \frac{-ln2}{ln\left[1 - \frac{r_i}{\sqrt{3}} CV_i \sqrt{\frac{n_i+1}{n_i-1}}\right]}$, where r_i and CV_i denote the i^{th} sample correlation coefficient between the observations and their ranks and the coefficient of variation respectively.

2. Estimates of the Parameters: Method Moments by Teimouri and Gupta (2013)

- Using this estimate of $\hat{\beta}_{itg}$, the estimate of $\hat{\alpha}_{itg}$ is $\hat{\alpha}_{itg} = \dfrac{\bar{m}_1}{\Gamma\left(1 + \frac{1}{\hat{\beta}_{itg}}\right)}$. As in section 2.2 we estimate the common value α of α_1 and α_2 under the null hypothesis as a weighted average of $\hat{\alpha}_{itg}$ but using $\hat{\beta}_{itg}$ instead of $\hat{\beta}_{ic}$ as $\hat{\alpha}_{tg} = \sum_{i=1}^{2} w_i \hat{\alpha}_{itg} / \sum_{i=1}^{2} w_i$, where $w_i = \dfrac{n_i}{\hat{V}_{itg}}$, $i = 1, 2$, $\hat{V}_{itg} = \hat{\alpha}_{itg}^2 \left[\Gamma\left(1 + \dfrac{2}{\hat{\beta}_{itg}}\right) - \left\{\Gamma\left(1 + \dfrac{1}{\hat{\beta}_{itg}}\right)\right\}^2\right]$. Denote these method of moments estimates by $\hat{\delta}_{tg} = \left(\hat{\alpha}_{tg}, \hat{\beta}_{1tg}, \hat{\beta}_{2tg}\right)$.

- Let $\hat{l}_1$ and $\hat{l}_0$ be the maximized log-likelihood under the alternative and the null hypothesis respectively. Then the likelihood ratio test statistic is $LR = 2\left(\hat{l}_1 - \hat{l}_0\right)$; which, under the null hypothesis, follows a χ^2 distribution with 1 degree of freedom.

Suppose the alternative hypothesis is represented by $\alpha_i = \alpha + \phi_i$, $i = 1, 2$, with $\phi_2 = 0$. Then the null hypothesis $H_0 : \alpha_1 = \alpha_2$ can equivalently be written as $H_0 : \phi_1 = 0$ with α, β_1 and β_2 treated as nuisance parameters. With this reparameterization, the log-likelihood can then be written as

$$l = \sum_{i=1}^{2} \left[n_i \log \left(\frac{\beta_i}{\alpha + \phi_i} \right) + (\beta_i - 1) \left\{ \sum_{j=1}^{n_i} \log \left(y_{ij} \right) - n_i \log \left(\alpha + \phi_i \right) \right\} - \frac{\sum_{j=1}^{n_i} y_{ij}^{\beta_i}}{(\alpha + \phi_i)^{\beta_i}} \right].$$

3. The Tests: C(α) Tests

- Now, let $\phi = \phi_1$ and $\delta = (\alpha, \beta_1, \beta_2)'$ and define $\psi = \frac{\partial l}{\partial \phi}\Big|_{\phi=0}$, $\gamma_1 = \frac{\partial l}{\partial \alpha}\Big|_{\phi=0}$, $\gamma_2 = \frac{\partial l}{\partial \beta_1}\Big|_{\phi=0}$, and $\gamma_3 = \frac{\partial l}{\partial \beta_2}\Big|_{\phi=0}$.
- Then the $C(\alpha)$ statistic is based on the adjusted score $S(\delta) = \psi - a_1\gamma_1 - a_2\gamma_2 - a_3\gamma_3$, where a_1, a_2 and a_3 are partial regression coefficient of ψ on γ_1, ψ on γ_2, and ψ on γ_3 respectively.

- The variance-covariance of $S(\delta)$ is $D - AB^{-1}A'$ and the regression coefficients $a = (a_1, a_2, a_3) = AB^{-1}$, where D is 1×1, A is 1×3 and B is 3×3 with elements

$$D = E\left[-\frac{\partial^2 l}{\partial \phi^2}\Big|_{\phi=0}\right], A_1 = E\left[-\frac{\partial^2 l}{\partial \phi \partial \alpha}\Big|_{\phi=0}\right],$$

$$A_2 = E\left[-\frac{\partial^2 l}{\partial \phi \partial \beta_1}\Big|_{\phi=0}\right], A_3 = E\left[-\frac{\partial^2 l}{\partial \phi \partial \beta_2}\Big|_{\phi=0}\right],$$

$$B_{11} = E\left[-\frac{\partial^2 l}{\partial \alpha^2}\Big|_{\phi=0}\right], B_{12} = B_{21} = E\left[-\frac{\partial^2 l}{\partial \alpha \partial \beta_1}\Big|_{\phi=0}\right],$$

$$B_{13} = B_{31} = E\left[-\frac{\partial^2 l}{\partial \alpha \partial \beta_2}\Big|_{\phi=0}\right], B_{22} = E\left[-\frac{\partial^2 l}{\partial \beta_1^2}\Big|_{\phi=0}\right],$$

$$B_{23} = B_{32} = E\left[-\frac{\partial^2 l}{\partial \beta_1 \partial \beta_2}\Big|_{\phi=0}\right] \text{ and } B_{33} = E\left[-\frac{\partial^2 l}{\partial \beta_2^2}\Big|_{\phi=0}\right].$$

- Derivation of the above elements based on the Weibull log-likelihood (3.1) are given in the Appendix.
- Substituting $\sqrt{n}$ (where $n = n_1 + n_2$) consistent estimate of δ in S, D, A and B, the $C(\alpha)$ statistic can be obtained as

$$C = S^2 \,/\, \left(D - AB^{-1}A'\right), \qquad (2)$$

which is approximately distributed as a chi-squared with 1 degree of freedom (Neyman, 1959; Neyman and Scott, 1966; Moran, 1970).

- If the maximum likelihood estimate $\hat{\delta}_{ml}$ of δ is used then $S = \psi$, and the $C(\alpha)$ statistic reduces to a score statistic (Rao, 1948)

$$C_{ml} = \psi^2 \,/\, (D - AB^{-1}A'). \tag{3}$$

- Further, two $C(\alpha)$ statistics are obtained from equation (3.2) by using $\hat{\delta}_{cr}$ and $\hat{\delta}_{tg}$ in all the expressions of S, D, A and B. Denote these $C(\alpha)$ statistics by C_{cr} and C_{tg} respectively. Each of the statistics C_{ml}, C_{cr} and C_{tg} is approximately distributed as a chi-squared with 1 degree of freedom.

Simulation Study: Levels

Table 3: Empirical level (%) of test statistics LR, C_{ml}, C_{cr} and C_{tg} for $\alpha_1 = \alpha_2 = \alpha$ and $\beta_2 = \beta + \beta_1$; based on 5000 iterations, $n_1 = n_2 = 5$, and nominal level $= 0.05$.

Statistic	(α, β_1)	β						
	(5, 3)	0.00	0.50	1.00	1.50	2.00	2.50	3.00
LR		3.2	3.3	3.3	3.8	3.9	3.6	3.5
C_{ml}		2.9	3.1	3.2	3.4	3.6	3.3	3.1
C_{cr}		4.1	4.4	4.3	4.2	4.5	4.8	4.4
C_{tg}		3.8	3.7	3.9	4.1	4.5	4.2	4.0
	(10, 6)							
LR		3.2	3.5	3.7	4.1	4.2	4.1	3.9
C_{ml}		3.0	3.1	3.4	3.4	3.8	3.6	3.3
C_{cr}		4.3	4.6	4.9	5.1	4.9	5.2	4.8
C_{tg}		3.7	4.1	4.4	4.7	4.5	4.8	4.4
	(15, 10)							
LR		3.4	3.8	4.1	4.2	4.3	4.5	4.4
C_{ml}		3.2	3.5	3.5	3.8	4.1	3.8	3.6
C_{cr}		4.5	4.7	4.8	5.0	5.4	5.2	5.4
C_{tg}		3.8	4.1	4.4	4.5	4.8	4.6	4.8

TABLE 4: Empirical level (%) of test statistics LR, C_{ml}, C_{cr} and C_{tg} for $\alpha_1 = \alpha_2 = \alpha$ and $\beta_2 = \beta + \beta_1$; based on 5000 iterations, $n_1 = n_2 = 10$, and nominal level $= 0.05$.

Statistic	(α, β_1)	β						
	(5, 3)	0.00	0.50	1.00	1.50	2.00	2.50	3.00
LR		3.6	3.8	4.2	4.4	4.4	4.4	4.5
C_{ml}		3.2	3.3	3.6	3.8	4.0	3.9	3.8
C_{cr}		4.1	4.6	5.1	5.1	5.2	5.1	5.4
C_{tg}		4.0	4.4	4.7	4.8	5.0	5.1	5.2
	(10, 6)							
LR		3.8	3.8	4.1	4.4	4.5	4.4	4.3
C_{ml}		3.4	3.5	4.0	4.0	4.3	4.4	4.1
C_{cr}		4.5	4.7	5.1	5.0	5.3	5.4	5.0
C_{tg}		4.1	4.3	4.5	4.7	4.7	5.1	4.8
	(15, 10)							
LR		3.4	3.9	4.1	4.6	4.6	4.7	4.4
C_{ml}		3.4	3.7	4.1	4.4	4.5	4.5	4.0
C_{cr}		4.5	5.2	5.3	5.2	5.3	5.4	5.6
C_{tg}		4.2	4.7	4.8	5.0	4.9	5.0	5.1

TABLE 5: Empirical level (%) of test statistics LR, C_{ml}, C_{cr} and C_{tg} for $\alpha_1 = \alpha_2 = \alpha$ and $\beta_2 = \beta + \beta_1$; based on 5000 iterations, $n_1 = n_2 = 20$, and nominal level = 0.05.

Statistic	(α, β_1)	β						
	(5, 3)	0.00	0.50	1.00	1.50	2.00	2.50	3.00
LR		3.7	3.7	4.3	4.5	4.7	4.7	4.7
C_{ml}		3.4	3.7	3.7	3.9	4.4	4.0	3.9
C_{cr}		4.5	5.0	5.2	5.2	5.1	5.4	5.2
C_{tg}		4.3	4.6	4.9	5.0	5.1	5.3	5.2
	(10, 6)							
LR		3.8	4.0	4.5	4.4	4.8	4.6	4.6
C_{ml}		3.6	3.6	3.9	4.1	4.6	4.5	4.4
C_{cr}		4.7	5.1	5.1	5.0	5.3	5.6	5.4
C_{tg}		4.2	4.6	5.1	4.8	5.0	5.2	5.0
	(15, 10)							
LR		3.9	4.2	4.4	4.7	5.1	4.7	4.8
C_{ml}		3.7	3.9	4.2	4.4	4.7	4.6	4.5
C_{cr}		4.5	4.8	5.4	5.5	5.5	5.4	5.7
C_{tg}		4.4	4.6	5.1	5.2	5.4	5.1	5.4

Simulation Study: Levels

Table 6: Empirical level (%) of test statistics LR, C_{ml}, C_{cr} and C_{tg} for $\alpha_1 = \alpha_2 = \alpha$ and $\beta_2 = \beta + \beta_1$; based on 5000 iterations, $n_1 = n_2 = 50$, and nominal level $= 0.05$.

Statistic	(α, β_1)	β						
	(5, 3)	0.00	0.50	1.00	1.50	2.00	2.50	3.00
LR		4.3	4.5	4.8	5.0	4.9	4.8	4.8
C_{ml}		3.9	4.2	4.2	4.2	4.6	4.4	4.2
C_{cr}		4.9	5.1	5.5	5.5	5.4	5.3	5.5
C_{tg}		4.6	4.8	5.0	5.2	5.0	5.3	5.1
	(10, 6)							
LR		4.3	4.6	4.7	4.7	5.1	4.9	4.9
C_{ml}		4.1	4.2	4.5	4.6	4.7	4.7	4.6
C_{cr}		5.1	5.1	5.5	5.4	5.3	5.6	5.6
C_{tg}		4.8	5.0	5.2	5.1	5.3	5.4	5.2
	(15, 10)							
LR		4.4	4.7	4.8	5.1	5.2	5.1	5.0
C_{ml}		4.4	4.5	4.5	4.7	4.7	4.8	4.8
C_{cr}		5.5	5.7	5.7	5.4	5.5	5.6	5.4
C_{tg}		5.2	5.2	5.5	5.4	5.5	5.4	5.0

Simulation Study: Powers

Table 7: Empirical power (%) of test statistics LR, C_{ml}, C_{cr} and C_{tg} for $\alpha_2 = \alpha_1 + \alpha$; based on 5000 iterations, $n_1 = n_2 = 5$, and nominal level $= 0.05$.

Statistic	$(\alpha_1, \beta_1, \beta_2)$	α								
	(5, 3, 4)	0.00	0.50	1.00	1.50	2.00	2.50	3.00	4.00	5.00
LR		3.1	3.8	6.0	10.6	16.7	24.8	36.8	57.6	81.9
C_{ml}		2.7	3.3	5.4	9.7	15.3	22.7	34.4	54.8	78.5
C_{cr}		4.0	4.8	7.0	11.9	17.5	26.1	38.5	59.4	83.7
C_{tg}		3.6	4.2	6.4	10.8	17.0	24.9	37.6	58.3	82.6
	(10, 3, 4)									
LR		3.1	3.7	5.6	10.3	16.5	24.7	36.3	57.4	81.4
C_{ml}		2.8	3.3	5.3	9.8	15.3	22.3	34.5	54.1	76.2
C_{cr}		4.1	4.7	6.7	11.5	18.0	27.1	38.9	60.6	83.6
C_{tg}		3.5	4.0	6.0	10.7	17.2	25.6	37.2	58.3	82.3
	(15, 3, 4)									
LR		3.2	3.9	5.7	10.5	16.8	24.3	35.9	56.8	82.5
C_{ml}		3.1	3.6	5.4	10.0	15.3	22.4	35.0	54.6	76.4
C_{cr}		4.5	5.1	6.9	11.8	17.8	27.1	38.4	59.3	83.5
C_{tg}		3.6	4.2	6.1	11.0	17.0	25.8	37.8	58.0	82.6

Simulation Study: Powers

Table 8: Empirical power (%) of test statistics LR, C_{ml}, C_{cr} and C_{tg} for $\alpha_2 = \alpha_1 + \alpha$; based on 5000 iterations, $n_1 = n_2 = 10$, and nominal level $= 0.05$.

Statistic	$(\alpha_1, \beta_1, \beta_2)$	α								
	(5, 3, 4)	0.00	0.50	1.00	1.50	2.00	2.50	3.00	4.00	5.00
LR		4.0	4.9	7.2	11.7	17.9	26.0	39.1	60.8	85.3
C_{ml}		3.3	4.1	6.3	10.9	16.7	24.6	36.9	58.4	82.5
C_{cr}		4.9	5.7	8.0	12.9	19.0	27.3	40.4	62.2	87.1
C_{tg}		4.6	5.5	7.7	12.5	18.3	26.6	39.8	61.7	86.0
	(10, 3, 4)									
LR		3.9	4.8	6.9	11.7	18.1	26.5	39.4	61.1	85.9
C_{ml}		3.8	4.5	6.5	11.0	16.8	25.1	37.2	58.7	82.7
C_{cr}		4.8	5.6	7.7	12.6	19.5	28.0	40.9	62.9	87.7
C_{tg}		4.4	5.1	7.2	12.3	18.7	27.4	40.0	62.6	87.0
	(15, 3, 4)									
LR		4.1	4.9	6.9	11.4	17.7	24.4	39.3	60.7	84.6
C_{ml}		3.9	4.5	6.4	10.8	16.3	23.7	35.0	58.6	82.7
C_{cr}		4.9	5.6	7.7	12.1	19.0	26.8	41.5	63.8	88.8
C_{tg}		4.2	5.1	7.1	12.0	18.2	25.6	40.4	62.3	86.6

Simulation Study: Powers

Table 9: Empirical power (%) of test statistics LR, C_{ml}, C_{cr} and C_{tg} for $\alpha_2 = \alpha_1 + \alpha$; based on 5000 iterations, $n_1 = n_2 = 20$, and nominal level $= 0.05$.

Statistic	$(\alpha_1, \beta_1, \beta_2)$	α								
	(5, 3, 4)	0.00	0.50	1.00	1.50	2.00	2.50	3.00	4.00	5.00
LR		4.2	5.3	7.6	12.7	19.8	28.9	42.3	63.8	88.1
C_{ml}		3.5	4.6	6.9	11.7	18.2	27.0	40.5	62.3	86.4
C_{cr}		5.1	6.3	8.7	13.9	22.6	32.0	44.4	68.1	91.8
C_{tg}		4.8	6.0	8.3	13.4	21.1	30.4	42.9	65.1	89.4
	(10, 3, 4)									
LR		4.4	5.5	7.7	13.2	22.3	32.6	45.3	67.4	92.2
C_{ml}		3.8	4.8	6.9	12.3	19.4	29.5	42.5	64.6	89.9
C_{cr}		5.1	6.3	8.5	14.1	25.9	36.2	49.1	73.3	96.4
C_{tg}		4.8	5.7	8.0	13.4	24.1	34.1	46.5	69.3	93.1
	(15, 3, 4)									
LR		4.3	5.4	7.5	13.1	23.9	34.0	46.4	68.3	93.8
C_{ml}		4.2	5.1	7.1	12.5	20.1	30.8	43.7	65.7	91.6
C_{cr}		5.5	6.5	8.7	14.6	27.8	38.1	50.9	75.4	98.3
C_{tg}		5.0	5.9	8.0	13.8	25.8	36.0	47.7	70.4	94.6

Simulation Study: Powers

Table 10: Empirical power (%) of test statistics LR, C_{ml}, C_{cr} and C_{tg} for $\alpha_2 = \alpha_1 + \alpha$; based on 5000 iterations, $n_1 = n_2 = 50$, and nominal level $= 0.05$.

Statistic	$(\alpha_1, \beta_1, \beta_2)$	α								
	(5, 3, 4)	0.00	0.50	1.00	1.50	2.00	2.50	3.00	4.00	5.00
LR		4.6	5.9	8.0	13.4	23.2	35.0	48.9	75.4	100
C_{ml}		4.1	5.5	7.5	12.7	22.0	33.7	47.1	73.2	100
C_{cr}		5.5	6.7	8.7	14.2	24.7	36.9	52.0	78.4	100
C_{tg}		4.9	6.2	8.1	13.8	23.9	35.8	50.3	77.0	100
	(10, 3, 4)									
LR		4.6	5.9	7.9	14.7	25.9	38.0	52.6	79.8	100
C_{ml}		4.5	5.8	7.6	13.9	24.1	34.9	49.1	76.4	100
C_{cr}		5.5	6.9	8.7	15.8	29.4	41.1	57.9	84.2	100
C_{tg}		5.1	6.4	8.4	15.3	27.9	39.8	55.5	81.6	100
	(15, 3, 4)									
LR		4.6	6.1	8.0	14.1	23.6	35.3	48.7	74.8	99.7
C_{ml}		4.4	5.7	7.5	13.6	22.9	33.2	46.4	71.7	98.8
C_{cr}		5.6	7.0	8.8	15.7	26.0	37.7	53.1	78.9	100
C_{tg}		5.2	6.5	8.3	15.0	24.9	36.1	51.7	78.2	100

SIMULATION STUDY: POWERS

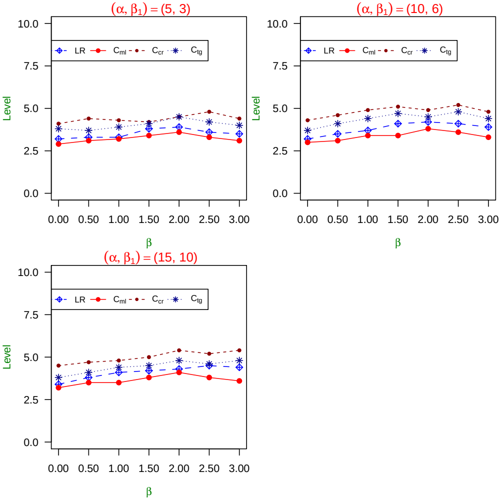

FIGURE 1: Plots of empirical levels for $n_1 = n_2 = 5$

Simulation Study: Powers

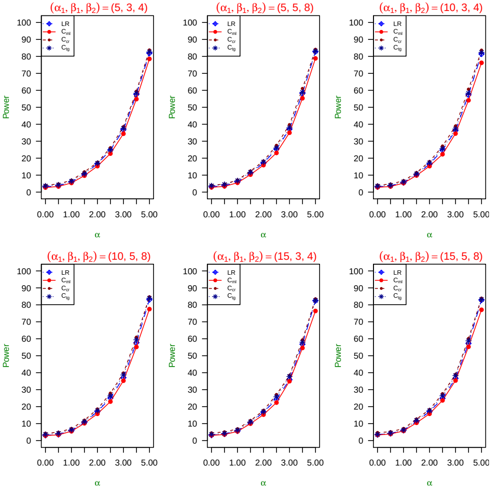

FIGURE 2: Power curves for $n_1 = n_2 = 5$