Multivariate data with block compound symmetry covariance structure

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- Hotelling's T^2 statistic is the conventional method:

$$T^2 = n(\bar{X} - \mu_0)' S^{-1} (\bar{X} - \mu_0) \sim T^2_{p,n-1}$$

where $\bar{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ is a sample mean and
$S = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})(X_i - \bar{X})'$ is a sample covariance matrix.

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- Problem of many time points:
  - Number of elements of Σ grows quickly.
  - Estimability and stability of the estimators requires a lot of observations.
  - Solution: a simpler variance structure keeps the number of unknown parameters reasonable.
- In practical applications patterned covariance matrices are of great significance. Several authors have assumed that

$$\Sigma = \sigma^2 R(\rho)$$

where σ^2 is the scale parameter and the patterned correlation matrix $R(\rho)$ is a function of the correlation scalar/vector parameter.

Patterned covariance matrix

- Wilks (1946): considered the **compound symmetry** (also known as **intraclass** or **uniform**) covariance structure when dealing with measurements on k equivalent psychological tests

$$R(\rho) = (1-\rho)I + \rho\mathbf{1}\mathbf{1}',$$

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$$R(\rho) = (1-\rho)I + \rho \mathbf{1}\mathbf{1}',$$

  where $-(p-1)^{-1} < \rho < 1$.
- For testing the hypothesis we cannot employ usual Hotelling T^2 test, because it uses only unstructured covariance matrix, and not the special one.

Historical remarks

- The problem of mean test with CS covariance structure was first considered by Geisser (1963), who arrived to the test statistic composed of a linear combination of two independent F-distributions.

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- Press (1967) considered Behrens-Fisher problem with CS covariance structures. His solution uses product of independent beta-distributions.
- Arnold (1973) considered testing problems in block compound symmetry covariance setting. He proposed the orthogonalization of the problem, and suggested testing by a product of independent beta-variates.

Two-level multivariate data

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Two-level multivariate data

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- These are called doubly multivariate or two-level multivariate data.
- Example: an investigator measured the mineral content of bones (radius, humerus and ulna) by photon absorptiometry to examine whether dietary supplements would slow bone loss in 25 older women. Measurements were recorded for three bones ($q = 3$) on the dominant and non-dominant sides ($p = 2$).

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  - let $\operatorname{Cov}\left(X_j^*, X_k^*\right) = \Sigma_1 \ \forall\, j \neq k$,
  - then $X \sim N_{pq}(\mu, \Gamma)$, where

$$\Gamma = \begin{pmatrix} \Sigma_0 & \Sigma_1 & \ldots & \Sigma_1 \\ \Sigma_1 & \Sigma_0 & \ldots & \Sigma_1 \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_1 & \Sigma_1 & \ldots & \Sigma_0 \end{pmatrix} = I_p \otimes (\Sigma_0 - \Sigma_1) + J_p \otimes \Sigma_1$$

(we need $\Sigma_0 - \Sigma_1 > 0, \Sigma_0 + (p-1)\Sigma_1 > 0$ for $\Gamma > 0$).

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  - one-sample test
  - paired samples test
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- $\underset{\sim}{X} = (X_1, \dots, X_n) = \left({\underset{\sim}{X^*_{\bullet 1}}}', \dots, {\underset{\sim}{X^*_{\bullet p}}}'\right)'$;
- $S = \frac{1}{n-1} \underset{\sim}{X} Q_n \underset{\sim}{X}' = \begin{pmatrix} S_{11} & S_{12} & \dots & S_{1p} \\ S_{21} & S_{22} & \dots & S_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ S_{p1} & S_{p2} & \dots & S_{pp} \end{pmatrix}$,

  where $S_{ij} = \frac{1}{n-1} \underset{\sim}{X^*_{\bullet i}} Q_n {\underset{\sim}{X^*_{\bullet j}}}'$;

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- It is natural to use the following estimators of variances and covariances:

$$\widehat{\Sigma}_0 = \frac{1}{p}\sum_{i=1}^{p} S_{ii}, \quad \widehat{\Sigma}_1 = \frac{1}{p(p-1)} \sum_{\substack{i=1 \\ i \neq j}}^{p} \sum_{j=1}^{p} S_{ij} \,.$$

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- The unbiased estimator of Γ is then

$$\widehat{\Gamma} = I_p \otimes \left(\widehat{\Sigma}_0 - \widehat{\Sigma}_1\right) + J_p \otimes \widehat{\Sigma}_1.$$

One-sample test

- $\widehat{\Gamma} = I_p \otimes \left(\widehat{\Sigma}_0 - \widehat{\Sigma}_1\right) + J_p \otimes \widehat{\Sigma}_1$ does not follow Wishart distribution $\rightarrow$ we cannot use standard T^2 test.

One-sample test

- $\widehat{\Gamma} = I_p \otimes \left(\widehat{\Sigma}_0 - \widehat{\Sigma}_1\right) + J_p \otimes \widehat{\Sigma}_1$ does not follow Wishart distribution $\rightarrow$ we cannot use standard T^2 test.
- Since H_0 is equivalent to $H_0: \; Z\mu = Z\mu_0$ for any non-singular matrix Z, we propose to use $Z = H_p \otimes I_q$, where H_p is a $p \times p$ orthogonal matrix with the first row proportional to vector of 1's.

One-sample test

- Then, we have

$$Y = ZX \sim N_{pq}\left(Z\mu, \Omega\right),$$

where

$$\Omega = Z\Gamma Z' = \begin{pmatrix} \Sigma_0 + (p-1)\Sigma_1 & 0 \\ 0 & I_{p-1} \otimes \left(\Sigma_0 - \Sigma_1\right) \end{pmatrix}.$$

One-sample test

- Neither the estimator $\widehat{\Omega} = \begin{pmatrix} \widehat{\Sigma}_0+(p-1)\widehat{\Sigma}_1 & 0 \\ 0 & I_{p-1}\otimes\left(\widehat{\Sigma}_0-\widehat{\Sigma}_1\right) \end{pmatrix}$ does not have a Wishart distribution.

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- It holds:

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- It holds:

Theorem

Distributions of

$$(n-1)(p-1)\left(\widehat{\Sigma}_0 - \widehat{\Sigma}_1\right),$$
$$(n-1)\left(\widehat{\Sigma}_0 + (p-1)\widehat{\Sigma}_1\right)$$

are independent, and

$$(n-1)(p-1)\left(\widehat{\Sigma}_0 - \widehat{\Sigma}_1\right) \sim W_q\left((n-1)(p-1), \Sigma_0 - \Sigma_1\right),$$

$$(n-1)\left(\widehat{\Sigma}_0 + (p-1)\widehat{\Sigma}_1\right) \sim W_q\left(n-1, \Sigma_0 + (p-1)\Sigma_1\right),$$

One-sample test

- Now, we have

$$Y_1, \ldots, Y_n \sim N_{pq}\left(Z\mu, \Omega\right),$$

and

$$\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \sim N_{pq}\left(Z\mu, \frac{1}{n}\Omega\right).$$

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- Denoting $Z\mu = \delta$ we consider the vectors $\overline{Y}$ and δ be partitioned in p q-dimensional subvectors as

$$\overline{Y} = \begin{pmatrix} \overline{Y}^*_{\bullet 1} \\ \vdots \\ \overline{Y}^*_{\bullet p} \end{pmatrix}, \quad \text{and} \quad \delta = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_p \end{pmatrix}.$$

One-sample test

- Since Ω is block-diagonal with $q \times q$ blocks, the corresponding q-dimensional parts of the sample mean $\overline{Y}^*_{\bullet j}$ are independent and it holds

$$\overline{Y}^*_{\bullet 1} \sim N_q\left(\delta_1, \frac{1}{n}\left(\Sigma_0 + (p-1)\Sigma_1\right)\right)$$
$$\overline{Y}^*_{\bullet j} \sim N_q\left(\delta_j, \frac{1}{n}\left(\Sigma_0 - \Sigma_1\right)\right),\ j = 2, \ldots, p.$$

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$$\overline{Y}^*_{\bullet j} \sim N_q\left(\delta_j, \frac{1}{n}\left(\Sigma_0 - \Sigma_1\right)\right),\ j = 2, \ldots, p.$$

- Then

$$\overline{\overline{Y}}^*_2 = \frac{1}{p-1}\sum_{j=2}^{p} \overline{Y}^*_{\bullet j} \sim N_q\left(\overline{\delta}_2, \frac{1}{n(p-1)}\left(\Sigma_0 - \Sigma_1\right)\right),$$

where $\overline{\delta}_2 = \left(\delta_2 + \cdots + \delta_p\right)/(p-1)$.

One-sample test

- The means are independent of the variance matrices estimators, under H_0 we have two independent T^2 statistics

$$n\left(\overline{Y}^{*}_{\bullet 1} - \delta_{01}\right)'\left(\widehat{\Sigma}_0 + (p-1)\widehat{\Sigma}_1\right)^{-1}\left(\overline{Y}^{*}_{\bullet 1} - \delta_{01}\right) \sim T^2_{q,n-1}\,,$$

$$n(p-1)\left(\overline{\overline{Y}}^{*}_{2} - \overline{\delta}_{02}\right)'\left(\widehat{\Sigma}_0 - \widehat{\Sigma}_1\right)^{-1}\left(\overline{\overline{Y}}^{*}_{2} - \overline{\delta}_{02}\right) \sim T^2_{q,(n-1)(p-1)}\,.$$

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$$n(p-1)\left(\overline{\overline{Y}}^*_2 - \overline{\delta}_{02}\right)' \left(\widehat{\Sigma}_0 - \widehat{\Sigma}_1\right)^{-1} \left(\overline{\overline{Y}}^*_2 - \overline{\delta}_{02}\right) \sim T^2_{q,(n-1)(p-1)}\,.$$

- A natural test statistics is the convolution of these two. We call it block T^2.

One-sample test

- It is of the form:

$$BT^2 = n\left(\overline{X} - \mu_0\right)' Z' \begin{pmatrix} \left(\widehat{\Sigma}_0 + (p-1)\widehat{\Sigma}_1\right)^{-1} & 0 \\ 0 & P_{p-1} \otimes \left(\widehat{\Sigma}_0 - \widehat{\Sigma}_1\right)^{-1} \end{pmatrix} Z\left(\overline{X} - \mu_0\right) \sim T^2_{q,n-1} \oplus T^2_{q,(n-1)(p-1)}.$$

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$$\sim T^2_{q,n-1} \oplus T^2_{q,(n-1)(p-1)}.$$

- The critical values or p-values of the test can be obtained using the method of Dyer (1982).

Paired samples test

- Let us have random samples $y_1, \ldots, y_n$ and $x_1, \ldots, x_n$ of doubly multivariate data measured before and after a treatment on the same individual i. So,

$$y_i \sim N_{pq}(\mu_y,\ I_p \otimes (\Sigma_{y0} - \Sigma_{y1}) + J_p \otimes \Sigma_{y1}),$$

$$x_i \sim N_{pq}(\mu_x,\ I_p \otimes (\Sigma_{x0} - \Sigma_{x1}) + J_p \otimes \Sigma_{x1}).$$

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- y_i and x_i are correlated and have a multivariate normal distribution:

$$\begin{pmatrix} y \\ x \end{pmatrix} \sim N_{2uq}\left[\begin{pmatrix} \mu_y \\ \mu_x \end{pmatrix}, \begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix}\right],$$

where

$$\begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix} = \begin{bmatrix} I_p \otimes (\Sigma_{y0} - \Sigma_{y1}) + J_p \otimes \Sigma_{y1} & J_p \otimes W \\ J_p \otimes W & I_p \otimes (\Sigma_{x0} - \Sigma_{x1}) + J_p \otimes \Sigma_{x1} \end{bmatrix},$$

where where W is a $q \times q$ symmetric matrix.

Paired samples test

- We want to test the effect of the treatment, which can be reformulated as testing equality of means, or equivalently, as zero difference of the corresponding means, i.e.

$$H_0: \ \mu_y - \mu_x = 0 \quad \text{against} \quad H_1: \mu_y - \mu_x \neq 0$$

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- Denoting $\mu_d = \mathrm{E}(y - x) = \mu_y - \mu_x$ we have the hypothesis

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- To estimate $\mathsf{Cov}(y - x) = \Sigma_{yy} - \Sigma_{yx} - \Sigma_{xy} + \Sigma_{xx}$, we need the estimates of $q \times q$ matrices Σ_{y1}, Σ_{y0}, Σ_{x1}, Σ_{x0} and W.

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- Denote $d_i = y_i - x_i$ then $d_1, \ldots, d_n$ are independent and identically distributed (i.i.d) $N_{uq}(\mu_d; \Gamma)$, where

$$
\begin{aligned}
\Gamma = \mathsf{Cov}(d) &= \mathsf{Cov}(y-x) \\
&= \Sigma_{yy} - \Sigma_{yx} - \Sigma_{xy} + \Sigma_{xx} \\
&= I_p \otimes (\Gamma_0 - \Gamma_1) + J_p \otimes \Gamma_1,
\end{aligned}
$$

where

$$
\Gamma_0 = \Sigma_{y0} + \Sigma_{x0} - 2W,
$$

$$
\Gamma_1 = \Sigma_{y1} + \Sigma_{x1} - 2W.
$$

Paired samples test

- Applying the results for one-sample test to $d_1, \ldots, d_n$ with $\mu_0 = 0$, we obtain the test statistic of $H_0: \ \mu_d = 0$ against $H_1: \ \mu_d \neq 0$ to be

$$BT_d^2 = n\overline{d}' Z' \begin{pmatrix} \left(\widehat{\Gamma}_0 + (p-1)\widehat{\Gamma}_1\right)^{-1} & 0 \\ 0 & P_{p-1} \otimes \left(\widehat{\Gamma}_0 - \widehat{\Gamma}_1\right)^{-1} \end{pmatrix} Z\overline{d}$$
$$\sim T^2_{q,n-1} \oplus T^2_{q,(n-1)(p-1)}.$$

Two-sample test

- Let us have two independent random samples $U_1, \ldots, U_n \sim N_{pq}(\mu_U, \Gamma)$ and $V_1, \ldots, V_m \sim N_{pq}(\mu_V, \Gamma)$. We want to test

$$H_0: \ \mu_U = \mu_V \quad \text{against} \quad H_1: \ \mu_U \neq \mu_V.$$

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- Sample means $\overline{U}$ and $\overline{V}$ are independent of variance matrices estimators $S_1 = \frac{1}{n-1} \underset{\sim}{U} Q_n \underset{\sim}{U}'$ and $S_2 = \frac{1}{m-1} \underset{\sim}{V} Q_n \underset{\sim}{V}'$, and thus also independent of the pooled estimator

$$S^p = \frac{1}{n+m-2}\left((n-1)S_1 + (m-1)S_2\right).$$

Two-sample test

- We have two independent statistics

$$\overline{U} - \overline{V} \sim N_{pq}\left(\mu_U - \mu_V, \frac{n+m}{nm}\Gamma\right),$$

$$S^p \sim W_{pq}\left(n+m-2, \frac{1}{n+m-2}\Gamma\right).$$

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- We can use the estimators

$$\widehat{\Gamma}_0 = \frac{1}{p}\sum_{i=1}^{p} S_{ii}^p, \quad \widehat{\Gamma}_1 = \frac{1}{p(p-1)}\sum_{\substack{i=1 \\ i\neq j}}^{p}\sum_{j=1}^{p} S_{ij}^p .$$

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- Applying the Theorem, we get

$$(n+m-2)(p-1)\left(\widehat{\Gamma}_0 - \widehat{\Gamma}_1\right) \sim W_q\left((n+m-2)(p-1), \Gamma_0 - \Gamma_1\right)$$

$$(n+m-2)\left(\widehat{\Gamma}_0 + (p-1)\widehat{\Gamma}_1\right) \sim W_q\left(n+m-2, \Gamma_0 + (p-1)\Gamma_1\right).$$

Two-sample test

- Since estimators $\widehat{\Gamma}_0 - \widehat{\Gamma}_1$ and $\widehat{\Gamma}_0 + (p-1)\widehat{\Gamma}_1$ are based on S^p, they are independent of $\overline{U} - \overline{V}$.

Two-sample test

- Since estimators $\widehat{\Gamma}_0 - \widehat{\Gamma}_1$ and $\widehat{\Gamma}_0 + (p-1)\widehat{\Gamma}_1$ are based on S^p, they are independent of $\overline{U} - \overline{V}$.
- Using analogous procedure as in the one-sample case, we arrive to block T^2 test statistic

$$BT^2 = \frac{nm}{n+m} \left(\overline{U} - \overline{V}\right)' Z' \begin{pmatrix} \left(\widehat{\Gamma}_0 + (p-1)\widehat{\Gamma}_1\right)^{-1} & 0 \\ 0 & P_{p-1} \otimes \left(\widehat{\Gamma}_0 - \widehat{\Gamma}_1\right)^{-1} \end{pmatrix} Z \left(\overline{U} - \overline{V}\right) \sim T^2_{q,n+m-2} \oplus T^2_{q,(n+m-2)(p-1)}.$$

Three-level multivariate data

- The procedure could be used also for the-level multivariate data with doubly exchangeable covariance structure.

Three-level multivariate data

- The procedure could be used also for the-level multivariate data with doubly exchangeable covariance structure.
- $X_1, \ldots, X_n$ be a sample from $N_{spq}(\mu, \Gamma)$, where

$$
\begin{aligned}
\Gamma &= I_{sp} \otimes (U_0 - U_1) + I_s \otimes J_p \otimes (U_1 - W) + J_{sp} \otimes W = \\
&= \left[\begin{array}{cccc|cccc|c|cccc}
U_0 & U_1 & \cdots & U_1 & W & W & \cdots & W & \cdots & W & W & \cdots & W \\
U_1 & U_0 & \cdots & U_1 & W & W & \cdots & W & \cdots & W & W & \cdots & W \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
U_1 & U_1 & \cdots & U_0 & W & W & \cdots & W & \cdots & W & W & \cdots & W \\
\hline
W & W & \cdots & W & U_0 & U_1 & \cdots & U_1 & \cdots & W & W & \cdots & W \\
W & W & \cdots & W & U_1 & U_0 & \cdots & U_1 & \cdots & W & W & \cdots & W \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
W & W & \cdots & W & U_1 & U_1 & \cdots & U_0 & \cdots & W & W & \cdots & W \\
\hline
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\hline
W & W & \cdots & W & W & W & \cdots & W & \cdots & U_0 & U_1 & \cdots & U_1 \\
W & W & \cdots & W & W & W & \cdots & W & \cdots & U_1 & U_0 & \cdots & U_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
W & W & \cdots & W & W & W & \cdots & W & \cdots & U_1 & U_1 & \cdots & U_0
\end{array}\right]
\end{aligned}
$$

Three-level multivariate data

Lemma

Let $Z = \underset{s\times s}{C} \otimes \underset{p\times p}{C^*} \otimes I_q$ *where* C *and* C^* *are orthogonal matrices whose first rows are proportional to* $\mathbf{1}$*'s. Let* Γ *be a doubly exchangeable covariance matrix, then* $Z\Gamma Z'$ *is a diagonal matrix with blocks on diagonal as follows:*

$$Z\Gamma Z' = \mathit{Diag}\left(\Delta_3; \Delta_1; \ldots; \Delta_1; \Delta_2; \Delta_1; \ldots; \Delta_1; \ldots; \Delta_2; \Delta_1; \ldots; \Delta_1\right),$$

where

$$\begin{aligned}
\Delta_1 &= U_0 - U_1, \\
\Delta_2 &= U_0 + (p-1)\,U_1 - pW = (U_0 - U_1) + p\,(U_1 - W), \\
\Delta_3 &= U_0 + (p-1)\,U_1 + p\,(s-1)\,W = (U_0 - U_1) + p\,(U_1 - W) + spW.
\end{aligned}$$

Unbiased Estimators of Δ_1, Δ_2 and Δ_3

- Let $X = (X_1, X_2, \ldots, X_n)$ be the data matrix from $N_{spq}(\mu, \Gamma)$ with doubly exchangeable covariance matrix Γ.

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- $$S = \frac{1}{n-1} X Q_n X' \sim W_{spq}\left(n-1, \frac{1}{n-1}\Gamma\right)$$
- Since $\mathrm{E}(S) = \Gamma$, the unbiased estimators of Δ_1, Δ_2 and Δ_3 are

$$\begin{aligned}
\widehat{\Delta}_1 &= \frac{1}{s(p-1)} \mathrm{BTr}_q[(I_s \otimes Q_p \otimes I_q)S], \\
\widehat{\Delta}_2 &= \frac{1}{s-1} \mathrm{BTr}_q\left[(Q_s \otimes P_p \otimes I_q)\, S\right], \\
\widehat{\Delta}_3 &= \mathrm{BTr}_q[(P_s \otimes P_p \otimes I_q)S].
\end{aligned}$$

Distributions of $\widehat{\Delta}_1$, $\widehat{\Delta}_2$ and $\widehat{\Delta}_3$

Theorem

The estimators $\widehat{\boldsymbol{\Delta}}_1$, $\widehat{\boldsymbol{\Delta}}_2$ *and* $\widehat{\boldsymbol{\Delta}}_3$ *are mutually independent and*

$$
\begin{aligned}
(n-1)s(p-1)\widehat{\boldsymbol{\Delta}}_1 &\sim W_q\left((n-1)s(p-1), \Delta_1\right), \\
(n-1)(s-1)\widehat{\boldsymbol{\Delta}}_2 &\sim W_q\left((n-1)(s-1), \Delta_2\right), \\
(n-1)\widehat{\boldsymbol{\Delta}}_3 &\sim W_q\left((n-1), \Delta_3\right).
\end{aligned}
$$

Test statistic

- The hypothesis:

$$H_0: \ \mu = \mu_0 \quad \text{vs} \quad H_1: \ \mu \neq \mu_0$$

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- The test statistic

$$BT^2 = n(\overline{X} - \mu_0)' Z' G Z (\overline{X} - \mu_0) \sim T^2_{q,n-1} + T^2_{q,(n-1)(s-1)} + T^2_{q,(n-1)s(p-1)},$$

where

$$G = \boldsymbol{e}_{1,s}\boldsymbol{e}'_{1,s} \otimes \boldsymbol{e}_{1,p}\boldsymbol{e}'_{1,p} \otimes \widehat{\Delta}_3^{-1} + P_s^0 \otimes \boldsymbol{e}_{1,p}\boldsymbol{e}'_{1,p} \otimes \widehat{\Delta}_2^{-1} + P_s \otimes P_p^0 \otimes \widehat{\Delta}_1^{-1},$$

and

$$P_p^0 = \frac{1}{p-1} J_p^0,$$

where J_p^0 is matrix J_p where the elements of first row and first column are zero.

- Under H_0 the statistics

$$\sqrt{n}\left(\mathbf{1}'_{sp}\otimes I_q\right)Z\left(\overline{X}-\mu_0\right)\sim N_q\left(\mathbf{0};\ sp\,U_0\right),$$

$$S^*=\left(\mathbf{1}'_{sp}\otimes I_q\right)ZSZ'\left(\mathbf{1}_{sp}\otimes I_q\right)\sim W_q\left(n-1;\ \frac{sp}{n-1}U_0\right)$$

are independent.

Another approach

- Under H_0 the statistics

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  are independent.
- Then

$$nsp\left(\overline{X}-\mu_0\right)'Z'\left(P_{sp}\otimes S^{*-1}\right)Z\left(\overline{X}-\mu_0\right)\sim T^2_{q,n-1}.$$

Power simulation

- The hypothesis tested is $H_0 : \mu = \mathbf{0}$.

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- The (3×3)-dimensional variance-covariance matrices Σ_0 and Σ_1 are taken as

$$\Sigma_0 = \begin{pmatrix} 1.54 & 0.63 & 0.26 \\ 0.63 & 7.26 & -0.31 \\ 0.26 & -0.31 & 1.57 \end{pmatrix}, \quad \Sigma_1 = \begin{pmatrix} 0.29 & 1.03 & -0.11 \\ 1.03 & 3.65 & -0.17 \\ -0.11 & -0.17 & 0.31 \end{pmatrix}.$$

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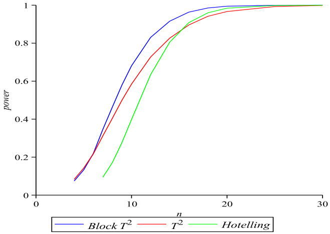
- Different real mean values μ are taken as $\mathbf{1}_{pq}$, $e_{1,p} \otimes \mathbf{1}_q$ and $e_{1,p} \otimes w$, where $w = (1, 2, \ldots, q)'$.

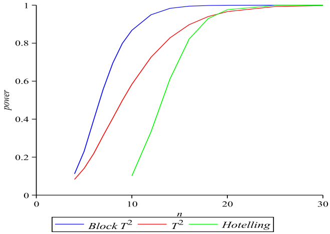
Power simulation

$\mu = \mathbf{1}_{pq}$

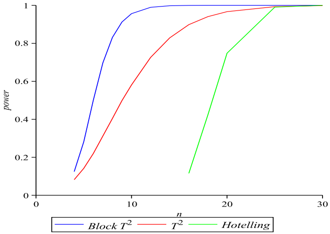
$p = 2$



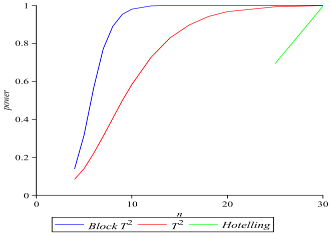
$p = 3$



$p = 5$

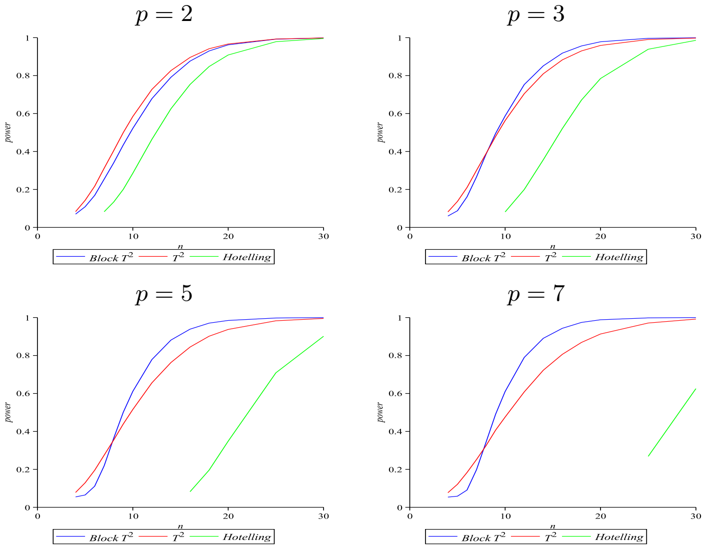


$p = 7$



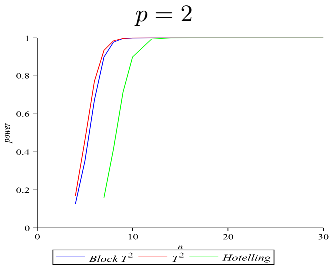
Power simulation

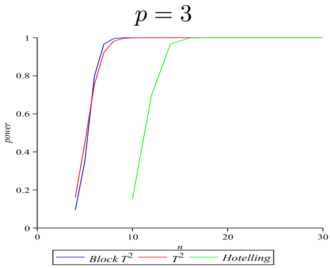
$\mu = e_{1,p} \otimes \mathbf{1}_q$

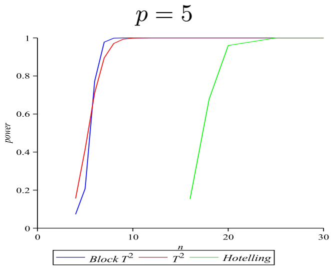


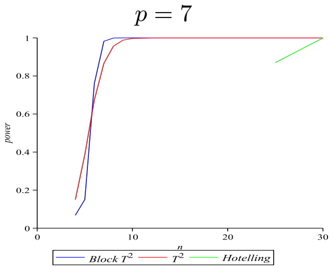
Power simulation

$\mu = e_{1,p} \otimes w$









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Thank you for your attention